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The dual Hahn q-polynomials in the lattice $x(s) = [s]_q[s+1]_q$ and the q-algebras $SU_q(2)$ and $SU_q(1,1)$

R Álvarez-Nodarse†§ and Yuri F Smirnov‡

† Departamento de Matemática Aplicada, Escuela Politécnica Superior, Universidad Carlos III de Madrid, Butarque 15, 28911, Leganés, Madrid

‡ Instituto de Física, Universidad Nacional Autónoma de México, Apdo Postal 20-364, México 20, DF and Institute of Nuclear Physics, Moscow State University, Moscow 119899, Russia

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Abstract. The dual q-Hahn polynomials in the non-uniform lattice $x(s) = [s]_q[s+1]_q$ are obtained. The main data for these polynomials are calculated (the square of the norm of the coefficients of the three-term recurrence relation, etc), as well as the lattice representation as a q-hypergeometric series. The connection with the Clebsch–Gordan coefficients of the quantum algebras $SU_q(2)$ and $SU_q(1,1)$ is also given.

1. Introduction

It is well known that the Lie groups representation theory plays a very important role in quantum theory and in special function theory. Group theory is an effective tool for the investigation of the properties of different special functions, moreover, it gives the possibility of unifying various special functions systematically. In a very simple and clear way, on the basis of group representation theory concepts, the special function theory was developed in the classical book of Vilenkin [1] and in the monographs of Vilenkin and Klimyk [2], which have an encyclopedic character.

In recent years, the development of the quantum inverse problem method [3] and the study of solutions of the Yang–Baxter equations [4] have given rise to the notion of quantum groups and algebras, which are, from the mathematical point of view, Hopf algebras [5]. They are of great importance for applications in quantum integrable systems, in quantum field theory, and statistical physics (see [6] and references contained therein). They have attracted much attention in quantum physics, especially after the introduction of the q-deformed oscillator [7, 8]. They have also been used for the description of the rotational and vibrational spectra of deformed nuclei [9–11] and diatomic molecules [12–14]. However, to apply them it is necessary to have a well developed theory of their representations. In quantum physics, for instance, the knowledge of the Clebsch–Gordan coefficients (3-j symbols), Racah coefficients (6-j symbols) and 9-j symbols [15] is crucial for applications because all the matrix elements of the physical quantities are proportional to them.

The present work represents the definite part of the investigations about the connection between different constructions of the Wigner–Racah algebras for the q-groups and q-algebras $SU_q(2)$ and $SU_q(1,1)$ and the orthogonal polynomials of discrete variables (see also

§ E-mail address: renato@dulcinea.uc3m.es

[16, 2 vol III, 17], as well as [18, 21–24, 26]). For a review of q-polynomials see [24, 26]. In [19] the properties of the Clebsch–Gordan coefficients (CGCs) of these two quantum algebras $SU_q(2)$ and $SU_q(1,1)$ and the q-analogue of the Hahn polynomials on the exponential lattice $x(s) = q^{2s}$ were considered in detail. In a similar way the Racah coefficients (6-j symbols) for such q-algebras have been connected with the Racah polynomials in the lattice $x(s) = [s]_q[s+1]_q$ [20, 21]. Recently, the q-analogues of the Kravchuk and Meixner polynomials on the non-uniform lattice $x(s) = q^{2s}$ were investigated (see [23] and references contained therein) in order to find their connection with the Wigner D-functions and Bargmann D-functions for the q-algebras $SU_q(2)$ and $SU_q(1,1)$, respectively.

To continue along this line it seems reasonable to investigate the interrelation between the CGCs for the quantum algebras $SU_q(2)$ and $SU_q(1,1)$ with q-analogues of the dual Hahn polynomials on the non-uniform lattice $x(s) = [s]_q[s+1]_q$. In order to solve this problem in sections 2 and 3 we discuss the properties of these q-polynomials, their explicit formula, and the representation in terms of the generalized q-hypergeometric functions ${}_3F_2$ [24] is obtained. In section 4, from the detailed analysis of the finite difference equations (2) for these q-polynomials, we deduce the relation between them and the CGCs for $SU_q(2)$, which help us to draw an analogy between the basic properties of the Clebsch–Gordan coefficients and these orthogonal q-polynomials. Since these coefficients are studied from a viewpoint of the theory of orthogonal polynomials, a group-theoretical interpretation arises for the basic properties of dual Hahn q-polynomials. In section 5 we find the relation between Clebsch–Gordan coefficients for the quantum algebra $SU_q(1,1)$ and the dual Hahn q-polynomials in two different ways; the first is as in the previous case, i.e. comparing the finite difference equation for the dual Hahn q-polynomials and the corresponding recurrence relation for the CGC, and the second uses the well known relation between the CGCs for the q-algebra $SU_q(1,1)$ and the CGCs for $SU_q(2)$.

Using the connection between the CGCs and these q-polynomials (see formulae (19) and (24) later) we find explicit formulae for the CGCs, as well as their representation in terms of the generalized q-hypergeometric functions ${}_3F_2$ or the basis hypergeometric series ${}_3\varphi_2$ [27].

In conclusion of this section it should be noted that a new approach to the investigation of the connection between the representation theory of algebras and the theory of orthogonal polynomials was suggested recently [28–32]. This allows one to also solve a new class of problems (so-called *quasi-exactly solvable problems*). This approach was extended to the q-difference equation in [33]. In [34] it was shown that a similar approach can also be formulated to study the classical orthogonal polynomials in the exponential lattice $x(s) = q^{2s}$ [24]. As for the quadratic lattice $x(s) = s(s+1)$ and the q-quadratic lattice $x(s) = [s]_q[s+1]_q$, the extension of this approach to such types of problem has not been found yet. Therefore, we apply here the standard method of [24] to the analysis of the dual Hahn q-polynomials in the q-quadratic lattice.

2. The dual Hahn q-polynomials in the non-uniform lattice $x(s) = [s]_q[s+1]_q$

Let us start with the study of some general properties of orthogonal polynomials of a discrete variable in non-uniform lattices. Let

$$\tilde{\sigma}(x(s))\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla Y(s)}{\nabla x(s)} + \frac{1}{2}\tilde{\tau}(x(s))\left[\frac{\Delta Y(s)}{\Delta x(s)} + \frac{\nabla Y(s)}{\nabla x(s)}\right] + \lambda Y(s) = 0$$
$$\nabla f(s) = f(s) - f(s-1) \qquad \Delta f(s) = f(s+1) - f(s) \tag{1}$$

be the finite difference equation of hypergeometric type for some lattice function $x(s)$, where $\nabla f(s) = f(s) - f(s-1)$ and $\Delta f(s) = f(s+1) - f(s)$ denote the backward and forward finite difference quotients, respectively. Here $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials in $x(s)$ of degree at most two and one, respectively, and λ is a constant. It is convenient (see [24, 25]) to rewrite (1) in the equivalent form

$$\begin{aligned}
&\sigma(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla Y(s)}{\nabla x(s)} + \tau(s)\frac{\Delta Y(s)}{\Delta x(s)} + \lambda Y(s) = 0 \\
&\sigma(s) = \tilde{\sigma}(x(s)) - \tfrac{1}{2}\tilde{\tau}(x(s))\Delta x(s-\tfrac{1}{2}) \qquad \tau(s) = \tilde{\tau}(x(s)).
\end{aligned} \tag{2}$$

It is known ([24, 25]) that for some special kind of lattices, solutions of (2) are orthogonal polynomials of a discrete variable, in other words they satisfy the orthogonality relation

$$\sum_{s_i=a}^{b-1} P_n(x(s_i))P_m(x(s_i))\rho(s_i)\Delta x(s_i - \tfrac{1}{2}) = \delta_{nm}d_n^2 \qquad s_{i+1} = s_i + 1 \tag{3}$$

where $\rho(s)$ is some non-negative function (weight function), i.e.

$$\rho(s_i)\Delta x(s_i - \tfrac{1}{2}) > 0 \qquad (a \leqslant s_i \leqslant b-1)$$

supported in a countable set of the real line (a, b) and such that

$$\begin{aligned}
&\frac{\Delta}{\Delta x(s-\frac{1}{2})}[\sigma(x)\rho(x)] = \tau(x)\rho(x) \\
&\sigma(s)\rho(s)x^l(s)x^k(s-\tfrac{1}{2})|_{s=a,b} = 0 \qquad \forall k, l \in \mathbb{N} \qquad (\mathbb{N} = \{0, 1, 2, \ldots\}).
\end{aligned}$$

Here d_n^2 denotes the square of the norm of the corresponding orthogonal polynomials.

They satisfy a three-term recurrence relation (TTRR) of the form

$$x(s)P_n(s) = \alpha_n P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s) \qquad P_{-1}(s) = 0 \qquad P_0(s) = 1. \tag{4}$$

The polynomial solutions of equation (2), denoted by $Y_n(x(s)) \equiv P_n(s)$, are uniquely determined, up to a normalizing factor B_n, by the difference analogue of the Rodrigues formula (see [24, p 66, equation (3.2.19)])

$$P_n(s) = \frac{B_n}{\rho(s)}\nabla_n^{(n)}[\rho_n(s)] \qquad \nabla_n^{(n)} = \frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}[\rho_n(s)] \tag{5}$$

where

$$x_m(s) = x\left(s + \frac{m}{2}\right) \qquad \rho_n(s) = \rho(n+s)\prod_{k=1}^{n}\sigma(s+k).$$

These solutions correspond to some values of λ_n—the eigenvalues of equation (2).

Let us to start with the study of the dual Hahn q-polynomials in the particular non-uniform lattice $x(s) = [s]_q[s+1]_q$, where $[n]_q$ denotes the so called *q-numbers*,

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

and q is, in general, a complex number $|q| \neq 1$.

We will use a result by Nikiforov *et al* [24, theorem 1, p 59] who established that for the lattice functions $x(s) = c_1q^{2s} + c_2q^{-2s} + c_3$, where c_1, c_2, and c_3 are some constants, equation (2) has a polynomial solution uniquely determined, up to a constant factor B_n, by (5). A simple calculation shows that our lattice $x(s) = [s]_q[s+1]_q$ belongs to this class. In fact we have

$$x(s) = \frac{q}{(q-q^{-1})^2}q^{2s} + \frac{q^{-1}}{(q-q^{-1})^2}q^{-2s} - \frac{q+q^{-1}}{(q-q^{-1})^2} \tag{6}$$

so for the lattice $x(s) = [s]_q[s+1]_q$ it is possible to obtain polynomial solutions of equation (2) (see the appendix), and these solutions are uniquely determined by the Rodrigues formula (5).

We are interested in constructing the polynomials in such a way that, in the limit $q \to 1$, they and all their principal attributes ($\sigma(s)$, $\tau(s)$, λ_n, $\rho(s)$, d_n^2, TTRR coefficients α_n, β_n, γ_n, etc) transform into the classical ones. We will call these polynomials *the q-analogue of the classical dual Hahn polynomials in the non-uniform lattice* $x(s) = [s]_q[s+1]_q$ and they will be denoted by $W_n^{(c)}(s, a, b)_q$ (see also [25]). In order to obtain these q-polynomials let us define the $\sigma(x(s))$ function such that in the limit $q \to 1$ it coincides with the $\sigma(s)$ for the classical polynomials, i.e.

$$\lim_{q\to 1} \sigma(x(s)) = (s-a)(s+b)(s-c).$$

Therefore we will choose the function $\sigma(s)$ as follows:

$$\sigma(s) = q^{s+c+a-b+2}[s-a]_q[s+b]_q[s-c]_q. \tag{7}$$

Following chapter III in [24] we can find *the main data* for the polynomials $W_n^{(c)}(s, a, b)_q$. The results of these calculations are provided in table 1 (see also the appendix). Everywhere, $\forall x \in \mathbb{N}$, we denote by $[x]_q!$ the q-factorial which satisfies the relation $[x+1]_q! = [x+1]_q[x]_q!$ and coincides with the $\tilde{\Gamma}_q(x)$ function introduced by Nikiforov *et al* ([24, p 67, equations (3.2.23)–(3.2.25)]). In general $\forall x \in \mathbb{R}$ the q-factorial is defined in terms of the standard $\Gamma_q(x)$ (see [24] or [26]) by the formula

$$\tilde{\Gamma}_q(x+1) = [x]_q! = q^{-x(x-1)/2}\Gamma_q(x+1).$$

It is clear that all characteristics of these q-polynomials coincide with the corresponding attributes for the classical dual Hahn polynomials (see [24, p 109, table 3.7]) in the limit $q \to 1$.

Table 1. Main data for the q-analogue of the Hahn polynomials $W_n^c(s, a, b)_q$.

$Y_n(s)$	$W_n^c(x(s), a, b)_q$	$x(s) = [s]_q[s+1]_q$
(a, b)	(a, b)	
$\rho(s)$	$\dfrac{q^{-s(s+1)}[s+a]_q![s+c]_q!}{[s-a]_q![s-c]_q![s+b]_q![b-s-1]_q!}$	
	$-\frac{1}{2} \leqslant a \leqslant< b-1$	$\lvert c\rvert < a+1$
$\sigma(s)$	$q^{s+c+a-b+2}[s-a]_q[s+b]_q[s-c]_q$	
$\tau(s)$	$-x(s) + q^{a-b+c+1}[a+1]_q[b-c-1]_q + q^{c-b+1}[b]_q[c]_q$	
λ_n	$q^{-n+1}[n]_q$	
B_n	$\dfrac{(-1)^n}{[n]_q!}$	
$\rho_n(s)$	$\dfrac{q^{-s(s+1+n)-n^2/2+n(a+c-b+\frac{3}{2})}[s+a+n]_q![s+c+n]_q!}{[s-a]_q![s-c]_q![s+b]_q![b-s-n-1]_q!}$	
d_n^2	$q^{-ab-bc+ac+a+c-b+1+2n(a+c-b)-n^2+5n}\dfrac{[a+c+n]_q!}{[n]_q![b-c-n-1]_q![b-a-n-1]_q!}$	
a_n	$\dfrac{q^{-\frac{3}{2}n(n-1)}}{[n]_q!}$	
α_n	$q^{3n}[n+1]_q$	
β_n	$q^{2n-b+c+1}[b-a-n+1]_q[a+c+n+1]_q$	
	$+q^{2n+2a+c-b+1}[n]_q[b-c-n]_q + [a]_q[a+1]_q$	
γ_n	$q^{n+3+2(c+a-b)}[n+a+c]_q[b-a-n]_q[b-c-n]_q$	

3. The explicit formula for the dual Hahn q-polynomials in the lattice $x(s) = [s]_q[s+1]_q$. The finite difference derivative formulae

3.1. The explicit formula for the dual Hahn q-polynomials

In order to obtain the explicit formula for the q-polynomials $W_n^{(c)}(s,a,b)_q$ we will use the Rodrigues formula (5). Firstly, notice that for the lattice $x(s) = [s]_q[s+1]_q$ verifies that the relation

$$x(s) - x(s-i) = [i]_q \nabla x(s-(i-1)/2) = [i]_q[2s-i+1]_q$$

holds. Then, by induction we can find the following expression for the operator $\nabla_n^{(n)}[f(s)]$:

$$\nabla_n^{(n)}[f(s)] = \sum_{m=0}^{n} \frac{(-1)^{n+m}[n]_q![2s-n+2m+1]_q}{[m]_q![n-m]_q!\prod_{k=0}^{m}[2s+m+1-k]_q} f(s-n+m).$$

Thus, the Rodrigues formula for the lattice $x(s) = [s]_q[s+1]_q$ takes the form (see also [24, p 69, equation (3.2.30)])

$$P_n(s) = B_n \sum_{m=0}^{n} \frac{(-1)^{n+m}[n]_q![2s-n+2m+1]_q}{[m]_q![n-m]_q!\prod_{k=0}^{m}[2s+m+1-k]_q} \frac{\rho_n(s-n+m)}{\rho(s)}. \tag{8}$$

Now using the main data for the $W_n^{(c)}(s,a,b)_q$ polynomials (table 1), equation (8) can be rewritten in the form

$$\begin{aligned} W_n^{(c)}(s,a,b)_q &= \frac{[s-a]_q![s+b]_q![s-c]_q![b-s-1]_q!}{q^{n^2/2-sn-n(a+c-b+\frac{5}{2})}[s+a]_q![s+c]_q!} \\ &\quad\times \sum_{m=0}^{n} \frac{(-1)^m[2s-n+2m+1]_q}{[m]_q![n-m]_q![2s+m+1]_q!} \\ &\quad\times \frac{q^{-m^2-2sm+nm-m}[2s+m-n]_q![s+a+m]_q![s+c+m]_q!}{[s-a-n+m]_q![s+b-n+m]_q![s-c-n+m]_q![b-s-m-1]_q!}. \end{aligned} \tag{9}$$

As a consequence of this representation we obtain the values of $W_n^{(c)}(s=a,a,b)_q$ and $W_n^{(c)}(s=b-1,a,b)_q$ at the ends of the interval of orthogonality (a,b):

$$W_n^{(c)}(s=a,a,b)_q = \frac{(-1)^n q^{-n^2/2+n(c-b+\frac{3}{2})}[b-a-1]_q![a+c+n]_q!}{[n]_q![a+c]_q![b-a-n-1]_q!} \tag{10}$$

$$W_n^{(c)}(s=b-1,a,b)_q = \frac{q^{-n^2/2+n(c+a+\frac{3}{2})}[b-a-1]_q![b-c-1]_q!}{[n]_q![b-c-n-1]_q![b-a-n-1]_q!}. \tag{11}$$

In order to find the representation of these polynomials in terms of q-hypergeometric functions we can follow [24] (chapter 3, section 3.11.2, p 135). Using the corresponding constants c_1, c_2, and c_3 (6) for the non-uniform lattice $x(s) = [s]_q[s+1]_q$ we obtain (see [24, equation (3.11.36), p 146]) the following:

$$\begin{aligned} W_n^{(c)}(x(s),a,b)_q &= \frac{(a-b+1|q)_n(a+c+1|q)_n}{q^{(n/2)(s+\frac{1}{2}(n-1))-\frac{1}{2}(c+a-b+1)}[n]_q!} \\ &\quad\times {}_3F_2\left(\begin{matrix}-n, a-s, a+s+1\\ a-b+1, a+c+1\end{matrix}; q, q^{\frac{1}{2}(b-c-n)}\right) \end{aligned} \tag{12}$$

where by definition

$${}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1|q)_k(a_2|q)_k(a_3|q)_k}{(b_1|q)_k(b_2|q)_k(q|q)_k} z^k$$

and

$$(\alpha|q)_n = \prod_{k=0}^{n-1}[\alpha+k]_q = [\alpha]_q[\alpha+1]_q \ldots [\alpha+n-1]_q = \frac{\tilde{\Gamma}_q(\alpha+n)}{\tilde{\Gamma}_q(\alpha)}.$$

3.2. The finite difference derivative formulae for the dual Hahn q-polynomials

To obtain the finite difference derivative formulae for these q-polynomials we will follow [24] (p 24, equation (2.2.9)). Firstly, notice the relation

$$\frac{\Delta P_n(s-\frac{1}{2})}{\Delta x(s-\frac{1}{2})} = \frac{\tilde{B}_{n-1}}{\tilde{\rho}(s)}\nabla_{n-1}^{(n-1)}[\rho_{n-1}(s)] = \frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_{n-1}(s)}[\tilde{\rho}_{n-1}(s)] = \tilde{P}_{n-1}(s)$$

where $\tilde{B}_{n-1} = -\lambda_n B_n$, $\tilde{\rho}(s) = \rho_1(s-\frac{1}{2})$, $\tilde{\rho}_{n-1}(s) = \rho_n(s-\frac{1}{2})$. In general, the polynomials $\tilde{P}_{n-1}(s)$ on the right-hand side of this equation are not the same as $P_n(s)$ (because they can have a different weight function). Since the following connection between weight functions holds for the q-analogue of dual Hahn polynomials in the lattice $x(s) = [s]_q[s+1]_q$,

$$\tilde{\rho}_{n-1}(s, a', b', c') = \rho_n(s-\tfrac{1}{2}, a, b, c) = \rho_{n-1}(s, a+\tfrac{1}{2}, b-\tfrac{1}{2}, c+\tfrac{1}{2}) \qquad (13)$$

we conclude that $\tilde{P}_{n-1}(s)$ coincides with the dual Hahn q-polynomial characterized by new parameters $a' = a+\frac{1}{2}$, $b' = b-\frac{1}{2}$, and $c' = c+\frac{1}{2}$. Then we obtain the following formula for the *finite difference derivative*:

$$W_n^{(c)}(s+\tfrac{1}{2}, a, b)_q - W_n^{(c)}(s-\tfrac{1}{2}, a, b)_q = q^{-3n+3}[2s+1]_q W_{n-1}^{(c+\frac{1}{2})}(s, a+\tfrac{1}{2}, b-\tfrac{1}{2})_q. \qquad (14)$$

The formula (14) will be called *the first differentiation formula* for the polynomials $W_n^{(c)}(s, a, b)_q$.

Now, if we change the parameters a, b and c and the variable s in $\rho(s)$ by $a' = a-\frac{1}{2}$, $b' = b+\frac{1}{2}$, $c' = c-\frac{1}{2}$, $s' = s-\frac{1}{2}$, we find

$$\tilde{\rho}_{n+1}(s-\tfrac{1}{2}, a', b', c') = q^{-2n+a+c-b+\frac{1}{4}}\rho_n(s, a, b, c). \qquad (15)$$

Then, from the Rodrigues formula (5),

$$\tilde{P}_{n+1}(s-\tfrac{1}{2}, a', b', c') = \frac{\tilde{B}_{n+1}}{\tilde{\rho}(s-\frac{1}{2}, a', b', c')}\nabla_{n+1}^{(n+1)}[\tilde{\rho}_{n+1}(s-\tfrac{1}{2}, a', b', c')]$$

and using equation (15) we obtain

$$\tilde{P}_{n+1}(s-\tfrac{1}{2}, a', b', c') = \frac{\tilde{B}_{n+1}q^{-2n+a+c-b+\frac{1}{4}}}{B_n\tilde{\rho}(s-\frac{1}{2}, a', b', c')}\frac{\nabla\rho(s, a, b, c)P_n(s)}{\nabla x(s)}.$$

As in the previous case, we notice that on the left-hand side of this equation the dual Hahn q-polynomials with a', b', c' parameters appear which are different from the corresponding parameters on the right-hand side. Namely, $a' = a-\frac{1}{2}$, $b' = b+\frac{1}{2}$, and $c' = c-\frac{1}{2}$. As a result the following formula for the finite difference derivative holds:

$$\begin{aligned} q^{2n-a-c+b}[n+1]_q[2s]_q W_{n+1}^{(c-\frac{1}{2})}(s-\tfrac{1}{2}, a-\tfrac{1}{2}, b+\tfrac{1}{2})_q & \\ = q^s[s-a]_q[s-c]_q[s+b]_q W_n^{(c)}(s-1, a, b)_q & \\ -q^{-s}[s+a]_q[s+c]_q[b-s]_q W_n^{(c)}(s, a, b)_q. & \qquad (16) \end{aligned}$$

Formula (16) will be called *the second differentiation formula* for the polynomials $W_n^{(c)}(s, a, b)_q$.

4. Clebsch–Gordan coefficients for the q-algebra $SU_q(2)$ and the dual Hahn q-polynomials

The quantum algebra $SU_q(2)$ is defined by three generators J_0, J_+, and J_- with the following properties (see [35, 36] and references therein):

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0]_q \qquad J_0^\dagger = J_0 \qquad J_\pm^\dagger = J_\mp.$$

Here we use the standard notation $[A, B] = AB - BA$ for the commutators, $[n]_q$ for q-numbers and $[2J_0]_q$ means the corresponding infinite formal series. Let D^{J_1} and D^{J_2} be two irreducible representations (IR) of the algebra $SU_q(2)$. The tensor product of two irreducible representations $D^{J_1} \otimes D^{J_2}$ can be decomposed into the direct sum of IR D^J components

$$D^{J_1} \otimes D^{J_2} = \sum_{J=|J_1-J_2|}^{J_1+J_2} \oplus D^J.$$

For the basis vectors of the IR D^J we have

$$|J_1 J_2, JM\rangle_q = \sum_{M_1, M_2} \langle J_1 M_1 J_2 M_2 | JM\rangle_q |J_1 M_1\rangle_q |J_2 M_2\rangle_q \tag{17}$$

where a symbol $\langle J_1 M_1 J_2 M_2 | JM\rangle_q$ denotes the CGCs for the quantum algebra $SU_q(2)$. In [35–38] it has been proved that these CGCs satisfy the following recurrence relation:

$$\begin{aligned}
(\{[J-M]_q[J+M]_q[J_1+J_2+J+1]_q[J_2-J_1+J]_q[J-J_2+J_1]_q[J_1+J_2-J+1]_q\}& \\
\times\{[2J+1]_q[2J-1]_q[2J]_q^2\}^{-1})^{\frac{1}{2}} \langle J_1 M_1 J_2 M_2 | J-1 M\rangle_q& \\
-\{(q^J[J+M+1]_q - q^{-J}[J-M+1]_q)([2J]_q[2J_2+2]_q& \\
-[2]_q[J_2+J_1-J+1]_q[J+J_1-J_2]_q)\}\{[2J+2]_q[2J]_q[2]_q\}^{-1}& \\
\times\langle J_1 M_1 J_2 M_2 | JM\rangle_q& \\
+(\{[J-M+1]_q[J+M+1]_q[J_1+J_2+J+2]_q[J_2-J_1+J+1]_q& \\
\times[J-J_2+J_1+1]_q[J_1+J_2-J]_q\}\{[2J+3]_q[2J]_q[2J+2]_q^2\}^{-1})^{\frac{1}{2}}& \\
\times\langle J_1 M_1 J_2 M_2 | J+1 M\rangle_q& \\
+\{(q^{J_2+M_1}[J_2+M_2+1]_q - q^{M_1-J_2}[J_2-M_2+1]_q)\}& \\
\times\{[2]_q\}^{-1}\langle J_1 M_1 J_2 M_2 | JM\rangle_q = 0.&
\end{aligned} \tag{18}$$

Comparing the difference equation (2) for the q-analogue of the dual Hahn polynomials $W_n^{(c)}(s, a, b)$ in the non-uniform lattice $x(s) = [s]_q[s+1]_q$ with the recurrence relation for CGCs, we conclude that CGCs $\langle J_1 M_1 J_2 M_2 | JM\rangle_q$ can be expressed in terms of the dual Hahn q-polynomials by the formula

$$\begin{aligned}
&(-1)^{J_1+J_2-J}\langle J_1 M_1 J_2 M_2 | JM\rangle_q = \frac{(\rho(s)\nabla x(s-\frac{1}{2}))^{\frac{1}{2}}}{d_n} W_n^{(c)}(x(s), a, b)_{q^{-1}} \\
&|J_1 - J_2| < M \qquad n = J_2 - M_2 \qquad s = J \qquad a = M \\
&c = J_1 - J_2 \qquad b = J_1 + J_2 + 1.
\end{aligned} \tag{19}$$

Here $\rho(x)$ and d_n denote the weight function and the normalization factor for the polynomials $W_n^{(c)}(x(s), a, b)_{q^{-1}}$, respectively. It should be noted that in (19) the parameter q is prescribed to the CGC $\langle J_1 M_1 J_2 M_2 | JM\rangle_q$, meanwhile the inverse parameter q^{-1} corresponds to the dual Hahn q-polynomial.

From the last expression and the orthogonality (3) of the $W_n^{(c)}(s,a,b)$ polynomials the orthogonality of the CGCs follows, i.e.

$$\sum_{J,M} \langle J_1 M_1 J_2 M_2 | J M\rangle_q \langle J_1 M_1' J_2 M_2' | J M\rangle_q = \delta_{M_1 M_1'} \delta_{M_2 M_2'}. \tag{20}$$

In the same way, we can show using (19) that the recursive relation (4) for the dual Hahn q-polynomials $W_n^{(c)}(s,a,b)$ is equivalent to the the *recursive relation* in M_1 and M_2 for the CGCs [35–38]

$$\begin{aligned} q^{-2}([J_2-M_2+1]_q[J_2+M_2]_q[J_1+M_1+1]_q[J_1-M_1]_q)^{\frac{1}{2}}\langle J_1 M_1+1 J_2 M_2-1|JM\rangle_q \\ +([J_2+M_2+1]_q[J_2-M_2]_q[J_1+M_1]_q[J_1-M_1+1]_q)^{\frac{1}{2}} \\ \times\langle J_1 M_1-1 J_2 M_2+1|JM\rangle_q \\ +(q^{-2M_1}[J_2+M_2+1]_q[J_2-M_2]_q+q^{2M_2}[J_1+M_1+1]_q[J_1-M_1]_q \\ +[M+\tfrac{1}{2}]_q^2-[J+\tfrac{1}{2}]_q^2)q^{-M_2+M_1-1}\langle J_1 M_1 J_2 M_2|JM\rangle_q = 0. \end{aligned} \tag{21}$$

The phase factor $(-1)^{J_1+J_2-J}$ in (19) was obtained by the comparison of the values of the $W_n^{(c)}(s,a,b)$ polynomials at the ends of the interval of orthogonality (see (10) and (11)) with the corresponding values of the CGCs at $J=M$ and $J=J_1+J_2+1$. Using relation (19) and the finite difference derivative formulae (14) and (16) we find the two recurrence relations for the CGCs:

$$\begin{aligned} q^{-J-1}\left(\frac{[J-M+1]_q[J_1+J_2+J+2]_q[J_2-J_1+J+1]_q[2J+2]_q}{[2J+3]_q[J_2-M_2]_q}\right)^{\frac{1}{2}} \\ \times\langle J_1 M_1 J_2 M_2|J+1 M\rangle_q \\ +\left(\frac{[J+M+1]_q[J_1+J_2-J]_q[J-J_2+J_1+1]_q[2J+2]_q}{[2J+1]_q[J_2-M_2]_q}\right)^{\frac{1}{2}} \\ \times\langle J_1 M_1 J_2 M_2|JM\rangle_q \\ = q^{(-J-J_2-M_2+M-1/2)}[2J+2]_q\langle J_1 M_1 J_2-\tfrac{1}{2} M_2+\tfrac{1}{2}|J+\tfrac{1}{2} M+\tfrac{1}{2}\rangle_q \end{aligned} \tag{22}$$

and

$$\begin{aligned} q^{-J}\left(\frac{[J-M]_q[J_1+J_2+J+1]_q[J_2-J_1+J]_q[2J]_q}{[2J-1]_q[J_2-M_2+1]_q}\right)^{\frac{1}{2}}\langle J_1 M_1 J_2 M_2|J-1 M\rangle_q \\ +\left(\frac{[J+M]_q[J_1+J_2-J+1]_q[J-J_2+J_1]_q[2J]_q}{[2J+1]_q[J_2-M_2+1]_q}\right)^{\frac{1}{2}}\langle J_1 M_1 J_2 M_2|JM\rangle_q \\ = q^{(-J-J_2-M_2+M-1)/2}[2J]_q\langle J_1 M_1 J_2+\tfrac{1}{2} M_2-\tfrac{1}{2}|J-\tfrac{1}{2} M-\tfrac{1}{2}\rangle_q. \end{aligned} \tag{23}$$

The formulae (22) and (23) can be obtained independently using the q-analogue of the quantum theory of angular momentum [35–38]. Let $T_\mu^{\frac{1}{2}}(2)$ be a tensor operator of rank $\frac{1}{2}$ acting on the variables J_2, M_2. If we calculate the matrix element $\langle J_1 M_1 J_2 M_2|T_\mu^{\frac{1}{2}}(2)|J_1' J_2'; J'M'\rangle_q$, on the one hand, using the Wigner–Eckart theorem for $SU_q(2)$ [35] we find that

$$\begin{aligned} \langle J_1 M_1 J_2 M_2|T_\mu^{\frac{1}{2}}(2)|J_1' J_2'; J'M'\rangle_q \\ = -\delta_{J_1,J_1'}\sum_{M_1' M_2'}\langle J_1' M_1' J_2' M_2'|J'M'\rangle_q \frac{\langle J_2' M_2' \frac{1}{2}\mu|J_2 M_2\rangle_q}{\sqrt{[2J_2+1]_q}}\langle J_2||T^{\frac{1}{2}}||J_2'\rangle_q. \end{aligned}$$

On the other hand, the application of the algebra of tensor operators [38] gives

$$\begin{aligned}
&\langle J_1M_1J_2M_2|T^{\frac{1}{2}}_{\mu}(2)|J_1'J_2';\,J'M'\rangle_q \\
&\quad= \sum_{J''M''}\langle J_1M_1J_2M_2|J''M''\rangle_q\langle J_1J_2;\,J''M''|T^{\frac{1}{2}}_{\mu}(2)|J_1'J_2';\,J'M'\rangle_q \\
&\quad= \sum_{J''M''}\langle J_1M_1J_2M_2|J''M''\rangle_q\frac{\langle J'M'\frac{1}{2}\mu|J''M''\rangle_q}{\sqrt{[2J''+1]_q}}(-1)^{J_1+J_2+J''+\frac{1}{2}} \\
&\qquad\times\sqrt{[2J''+1]_q[2J'+1]_q}\begin{Bmatrix} J_1 & J'' & J_2 \\ \frac{1}{2} & J_2' & J'\end{Bmatrix}_q\langle J_2||T^{\frac{1}{2}}||J_2'\rangle_q.
\end{aligned}$$

Putting in both of these equations $J' = J+\frac{1}{2}$, $M' = M+\frac{1}{2}$, $J_2' = J_2-\frac{1}{2}$, $J_1' = J_1$, $M_1' = M_1$, $M_2' = M_2+\frac{1}{2}$, $\mu = -\frac{1}{2}$ and taking into account that at such a choice of the angular momenta and their projections we obtain that only the values $M'' = M$, $J'' = J, J+1$ are possible. From this fact the relation (22) follows.

To obtain equation (23) we put $J' = J-\frac{1}{2}$, $M' = M-\frac{1}{2}$, $J_2' = J_2+\frac{1}{2}$, $J_1' = J_1$, $M_1' = M_1$, $M_2' = M_2-\frac{1}{2}$, $\mu = \frac{1}{2}$. All necessary quantities $\begin{Bmatrix} J_1 & J'' & J_2 \\ \frac{1}{2} & J_2' & J'\end{Bmatrix}_q$ and $\langle J_1M_1\frac{1}{2}\mu|JM\rangle_q$ are tabulated in [35, 36], respectively.

From relation (19) we also see that the dual Hahn q-polynomial with $n=0$ corresponds to the CGC with the maximal value of the projection of the angular momentum J_2, i.e. $M_2 = J_2$. For this reason it will be called *the backward way* (we start from $n=0$ and obtain the CGC at $M_2 = J_2$, for $n=1$ the CGC at $M_2 = J_2-1$, and so on). There exists another possibility corresponding to the inverse case, i.e. when the polynomial with $n=0$ is proportional to the CGC with the minimal value of $M_2 = -J_2$, this relation will be called *the forward way* (we start from $n=0$ and obtain the CGC at $M_2 = -J_2$, when $n=1$ we find the CGC at $M_2 = -J_2+1$, and so on). In fact, comparing the difference equation for the q-analogue of the dual Hahn polynomials $W_n^{(c)}(s,a,b)$ (2) with the recurrence relation for CGCs, we conclude that CGCs $\langle J_1M_1J_2M_2|JM\rangle_q$ can also be expressed in terms of the q-dual Hahn polynomials as follows:

$$\begin{gathered}
(-1)^{J_1+J_2-J}\langle J_1M_1J_2M_2|JM\rangle_q = \frac{\sqrt{\rho(s)\nabla x(s-\frac{1}{2})}}{d_n}W_n^{(c)}(x(s),a,b)_q \\
|J_1-J_2| < -M \qquad n = J_2+M_2 \qquad s = J \qquad a = -M \\
c = J_1-J_2 \qquad b = J_1+J_2+1.
\end{gathered} \tag{24}$$

Here, as earlier, $\rho(x)$ and d_n denote the weight function and the normalization factor for the polynomials $W_n^{(c)}(x(s),a,b)_q$, respectively.

Notice that if in the previous relation we provide the change of parameters $M_1 = -M_1$, $M_2 = -M_2$, $M = -M$, and $q = q^{-1}$ then the right-hand side of (24) coincides with the right-hand side of (19). Then, we can conclude that for the CGCs the following *symmetry property* holds:

$$(-1)^{J_1+J_2-J}\langle J_1-M_1J_2-M_2|J-M\rangle_{q^{-1}} = \langle J_1M_1J_2M_2|JM\rangle_q. \tag{25}$$

To conclude this section we provide table 2 in which the corresponding properties of the Hahn q-polynomials $h_n^{(\alpha,\beta)}(s,N)_q$ defined on the exponential lattice q^{2s} [19] (see also [24, 26]) and the dual Hahn q-polynomials $W_n^{(c)}(x(s),a,b)_q$ defined on the lattice $x(s) = [s]_q[s+1]_q$ are compared with the corresponding properties for the CGCs of the q-algebra $SU_q(2)$. This helps us to establish the inter-relation between these two types of orthogonal q-polynomials.

Table 2. CGCs and the q-analogue of Hahn polynomials.

$P_n(s)_q$	$\langle J_1M_1J_2M_2\|JM\rangle_q$
Finite difference equation (2) for the $W_n^{(c)}(x(s),a,b)_q$ and TTRR (4) for $h_n^{(\alpha,\beta)}(s,N)_q$	Recurrence relation (18) for the CGCs
Finite difference equation (2) for the $h_n^{\alpha,\beta}(s,N)_q$ and TTRR (4) for $W_n^{(c)}(x(s),a,b)_q$	Recurrence relation (21) for the CGCs
$\dfrac{\rho(s)}{d_n^2}$ in (19)	$\langle J_1M_1J_2J_2\|JM\rangle_q^2$
$\dfrac{\rho(s)}{d_n^2}$ in (24)	$\langle J_1M_1J_2-J_2\|JM\rangle_q^2$
Differentiation formulae (14) and (16) for $W_n^{(c)}(x(s),a,b)_q$	Recurrence relations (22) and (23) for the CGCs
Equivalence of relation (19) and (24)	Symmetry property (25) for the CGCs
Orthogonality relation (3)	Orthogonality relations (20)

Comparing the finite difference equation and the TTRR which the polynomials $h_n^{\alpha,\beta}(s,N)_q$ and $W_n^{(c)}(x(s),a,b)_q$ satisfy, we conclude

The finite difference equation (2) for the dual Hahn q-polynomials $W_n^{(c)}(x(s),a,b)_q$	$\longleftrightarrow$	Recurrence relation (4) for the q-Hahn polynomials $h_n^{(\alpha,\beta)}(s,N)_q$
Recurrence relation (4) for the dual Hahn q-polynomials $W_n^{(c)}(x(s),a,b)_q$	$\longleftrightarrow$	The finite difference equation (2) for the q-Hahn polynomials $h_n^{(\alpha,\beta)}(s,N)_q$

Moreover, since for the Hahn q-polynomials $h_n^{(\alpha,\beta)}(s,N)_q$ in the exponential lattice $x(s)=q^{2s}$ [19] the following relation holds (for the classical case see [24] and for the q-case see [19]):

$$(j_1m_1j_2m_2|jm)_{q^{-1}}=(-1)^s\sqrt{\frac{\rho(s)\Delta x(s-\frac12)}{d_n^2}}\,h_n^{\alpha\beta}(s,N)_q$$

where $s=j_2-m_2$, $N=j_1+j_2-m+1$, $\alpha=m-j_1+j_2$, $\beta=m+j_1-j_2$, and $n=j-m$. $\rho(x)$ and d_n denote the weight function and the normalization factor for the polynomials $h_n^{(\alpha,\beta)}(s,N)_q$ given by formulae

$$\rho(x)=q^{\frac12\alpha(\alpha+2N+2s-3)+\frac12\beta(\beta+2s-1)}\frac{[\alpha+N-2-1]![\beta+s]!}{[N-s-1]![s]!}$$

$$d_n^2=(q-q^{-1})^{2n}B_n^2\frac{[n]![\alpha+n]![\beta+n]![\alpha+\beta+N+n]!}{[N-n-1]![\alpha+\beta+n]![\alpha+\beta+2n+1]!}$$
$$\times q^{2\alpha+2N+N(N-1)+(N-1)(2\alpha+\beta+N)+\frac12\beta(\beta+1)+n(\alpha+\beta+2)}$$

where $B_n=(-1)^n/([n]!q^{2n}(q-q^{-1})^n)$. We obtain the following relation between Hahn q-polynomials $h_n^{(\alpha,\beta)}(s,N)_q$ and the dual Hahn q-polynomials $W_n^{(c)}(x(s),a,b)_q$:

$$(-1)^{s+n}q^{+(s-3)(\beta/2+\alpha)+3\alpha-2n\alpha-\frac32(n+s)+(n-s)(N-(\alpha+\beta)/2})h_n^{(\alpha,\beta)}(s,N)_q$$
$$=\frac{q^{2n}[s]_q![N-s-1]_q![n+\beta]_q!}{[n]_q![N-n-1]_q![s+\beta]_q!}W_s^{((\beta-\alpha)/2)}\left(t_n,\frac{\beta+\alpha}{2},\frac{\beta+\alpha}{2}+N\right)_q$$

$$\left(t_n = s_n(s_n+1) \qquad s_n = \frac{\beta+\alpha}{2}+n \qquad s,n=0,1,2,\ldots,N-1\right). \tag{26}$$

Observe that in the limit $q \to 1$ this relation takes the form of the classical relation between the classical Hahn and dual Hahn polynomials [24, p 76, equation (3.5.14)]).

5. The explicit formula for the CGCs. Its representation in terms of a basic hypergeometric function

5.1. The explicit formula for the Clebsch–Gordan coefficients of the $SU_q(2)$ quantum algebra

In order to obtain the explicit formula for the CGC $\langle J_1M_1J_2M_2|JM\rangle_q$ we will use the explicit expression for the dual Hahn q-polynomials (9) and equation (19), connecting them with CGCs. Providing some straightforward calculations we obtain the following general analytical formula to calculate the CGCs for the algebra $SU_q(2)$:

$$\begin{aligned}
&\langle J_1M_1J_2M_2|JM\rangle_q q^{-\frac{1}{2}(J(J+1)-J_1(J_1+1)+J_2(J_2+1))+(M+1)J_2+J(J_2-M_2)} \\
&\quad = (-1)^{J_1+J_2-J}\left(\frac{[J_2-M_2]_q![J_1-M_1]_q![J-M]_q![J_2+M_2]_q!}{[J+M]_q!}\right)^{\frac{1}{2}} \\
&\quad\times\left(\frac{[J+J_1+J_2+1]_q![J_2-J_1+J]_q![J_2+J_1-J]_q![2J+1]_q}{[J_1+M_1]_q![J_1-J_2+J]_q!}\right)^{\frac{1}{2}} \\
&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k q^{k^2+2Jk-(J_2-M_2-1)k}[J+J_1-J_2+k]_q![J+M+k]_q!}{[k]_q![2J+1+k]_q![J-M_1-J_2+k]_q![J-J_1+M_2+k]_q!} \\
&\quad\times\frac{[2J-J_2+M_2+k]_q![2J-J_2+M_2+2k+1]_q}{[J_2-M-2-k]_q![J+J_1+M_2+k+1]_q![J_1+J_2-J-k]_q!}.
\end{aligned} \tag{27}$$

5.2. Representation in terms of the basic hypergeometric function

In order to find the representation of the CGCs in terms of q-hypergeometric functions we can use the representation of the dual Hahn q-polynomials (12). Then, from formula (19) we obtain

$$\begin{aligned}
(-1)^{J_1+J_2-J}\langle J_1M_1J_2M_2|JM\rangle_{q^{-1}} &= \frac{\sqrt{\rho(s)[2s+1]_q}}{d_n}\,\frac{(a-b+1|q)_n(a+c+1|q)_n}{q^{n/2(s+\frac{1}{2}(n-1))-\frac{1}{2}(c+a-b+1)}[n]_q!} \\
&\quad\times {}_3F_2\left(\begin{matrix}-n, a-s, a+s+1\\ a-b+1, a+c+1\end{matrix}; q, q^{\frac{1}{2}(b-c-n)}\right)
\end{aligned} \tag{28}$$

where $|J_1-J_2|<M$, $n=J_2-M_2$, $s=J$, $a=M$, $c=J_1-J_2$, $b=J_1+J_2+1$, and $\rho(x)$ and d_n denote, as usual, the weight function and the normalization factor for the polynomials $W_n^{(c)}(x(s),a,b)_q$.

6. Clebsch–Gordan coefficients for the q-algebra $SU_q(1,1)$ and the dual Hahn q-polynomials

In the previous sections we have studied the connection between dual Hahn q-polynomials and the CGCs of the $SU_q(2)$ quantum algebra. Let us now study the connection between the dual Hahn polynomials and the Clebsch–Gordan coefficients of the quantum algebra $SU_q(1,1)$ (for a survey see [2, 39]). The quantum algebra $SU_q(1,1)$ is defined by three generators K_0, K_+ and K_- with the following properties [35]:

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -[2K_0]_q \qquad K_0^\dagger = K_0 \qquad K_\pm^\dagger = K_\mp.$$

Since this algebra is non-compact the IR can be classified in two series, the continuous and the discrete series of IR. In this work we will study the discrete case only, more precisely the *positive* discrete series D^{j+}. The basis vectors $|jm\rangle_q$ of the IR D^{j+} can be found from the *lowest weight vector* $|jj+1\rangle$ $(K_-|jj+1\rangle = 0)$ by the formula

$$|jm\rangle = \sqrt{\frac{[2j+1]_q!}{[j+m]_q![m-j-1]_q!}}K_+^{m-j-1}|jj+1\rangle.$$

Let D^{j_1+} and D^{j_2+} be two IR from the positive discrete series of the algebra $SU_q(1,1)$. The tensor product of these two IR, $D^{j_1+}\otimes D^{j_2+}$, can be decomposed into the direct sum of IR D^{j+} components

$$D^{j_1+}\otimes D^{j_2+} = \sum_{j=j_1+j_2+1}^{\infty} \oplus D^{j+}.$$

For the basis vectors of the IR D^{j+} we have

$$|j_1j_2, jm\rangle_q = \sum_{m_1,m_2} \langle j_1m_1j_2m_2|jm\rangle_q |j_1m_1\rangle_q |j_2m_2\rangle_q \tag{29}$$

where the symbols $\langle j_1m_1j_2m_2|jm\rangle_q$ denote the CGCs for the quantum algebra $SU_q(1,1)$. In [19] it was proved that these CGCs satisfy the following recurrence relation:

$$\begin{aligned}
&([m_2-j_2-1]_q[j_2+m_2]_q[m_1-j_1]_q[j_1+m_1+1]_q)^{\frac12}\langle j_1m_1+1j_2m_2-1|jm\rangle_q\\
&\quad +q^2([m_2-j_2]_q[j_2+m_2+1]_q[j_1+m_1]_q[m_1-j_1-1]_q)^{\frac12}\\
&\quad \times\langle j_1m_1-1j_2m_2+1|jm\rangle_q + (q^{-2m_1}[j_2+m_2+1]_q[m_2-j_2]_q\\
&\quad +q^{2m_2}[j_1+m_1+1]_q[m_1-j_1]_q + [j+\tfrac12]_q^2 - [m+\tfrac12]_q^2)q^{-m_2+m_1+1}\\
&\quad \times\langle j_1m_1j_2m_2|jm\rangle_q = 0.
\end{aligned} \tag{30}$$

Comparing the recurrence relation for the q-analogue of the dual Hahn polynomials $W_n^{(c)}(s,a,b)$ (4) with (30) for CGCs, we conclude that CGCs $\langle j_1m_1j_2m_2|jm\rangle_q$ can be expressed in terms of the dual Hahn q-polynomials by the formula

$$(-1)^{m-j-1}\langle j_1m_1j_2m_2|jm\rangle_q = \frac{\sqrt{\rho(s)\nabla x(s-\frac12)}}{d_n}W_n^{(c)}(x(s),a,b)_{q^{-1}} \tag{31}$$

$$n=m_1-j_1-1 \qquad s=j \qquad a=j_1+j_2+1 \qquad c=j_1-j_2 \qquad b=m.$$

We obtain the phase factor $(-1)^{m-j-1}$ comparing the values of the $W_n^{(c)}(s,a,b)$ polynomials at the ends of the interval (10) with the corresponding values of the CGCs. Now we can observe that if we provide the following substitution:

$$J_1 = \frac{m+j_1-j_2-1}{2} \qquad M_1 = \frac{m_1-m_2+j_1+j_2+1}{2} \qquad J=j$$

$$J_2 = \frac{m-j_1+j_2-1}{2} \qquad M_2 = \frac{m_2-m_1+j_1+j_2+1}{2} \qquad M=j_1+j_2+1$$

the right-hand sides of equations (19) and (31) become identical (see also [19]). This implies that for the CGCs for these two quantum algebras the following relation holds:

$$\langle J_1M_1J_2M_2|JM\rangle_{su_q(2)} = \langle j_1m_1j_2m_2|jm\rangle_{su_q(1,1)}. \tag{32}$$

Now we can obtain a general formula to calculate the CGCs for the $SU_q(1,1)$ algebra. Using the explicit expression for the q-analogue of the dual Hahn polynomials (9) and equation (31) we obtain

$$\begin{aligned}
&\langle j_1 m_1 j_2 m_2 | jm\rangle_q q^{-\frac{1}{2}(j(j+1)+j_1(j_1+1)-j_2(j_2+1))+(m-1)(j_1+1)+j(m_1-j_1-1)} \\
&\quad = (-1)^{m-j-1}\left(\frac{[j+m]_q![m-j-1]_q![m_2+j_2]_q!}{[j_1+m_1]_q!}\right)^{\frac{1}{2}} \\
&\quad\quad \times \left(\frac{[j-j_1-j_2-1]_q![j_2-j_1+j]_q![m_1-j_1-1]_q![m_2-j_2-1]_q![2j+1]_q}{[j+j_1+j_2+1]_q![j_1-j_2+j]_q!}\right)^{\frac{1}{2}} \\
&\quad\quad \times \sum_{k=0}^{\infty}\frac{(-1)^k q^{k^2+2jk-(m_1-j_1-2)k}[2j+j_1+1-m+k]_q![j-1+j_2+j+k+1]_q!}{[k]_q![2J+1+k]_q![m_1-j_1-1-k]_q![j-m_1-j_2+k]_q![m-j-1-k]_q!} \\
&\quad\quad \times \frac{[j+j_1-j_2+k]_q![2j+j_1-m_1+2k+2]_q}{[j-m_1+j_2+k]_q![j+j_1+m_2+k+1]_q!}.
\end{aligned} \tag{33}$$

Using formula (31) we find the following representation for the CGC of the $SU_q(1,1)$ quantum algebra in terms of the q-hypergeometric function:

$$\begin{aligned}
(-1)^{m-j-1}\langle j_1 m_1 j_2 m_2 | jm\rangle_{q^{-1}} &= \frac{\sqrt{\rho(s)[2s+1]_q}}{d_n}\,\frac{(a-b+1|q)_n(a+c+1|q)_n}{q^{n/2(s+\frac{1}{2}(n-1))-\frac{1}{2}(c+a-b+1)}[n]_q!} \\
&\quad \times {}_3F_2\left(\begin{matrix}-n, a-s, a+s+1\\ a-b+1, a+c+1\end{matrix}; q, q^{\frac{1}{2}(b-c-n)}\right)
\end{aligned} \tag{34}$$

where $n = m_1 - j_1 - 1$, $s = j$, $a = j_1 + j_2 + 1$, $b = m$, $c = j_1 - j_2$. We note that $\rho(x)$ and d_n denote the weight function and the normalization factor for the polynomials $W_n^{(c)}(x(s), a, b)_q$, respectively.

To conclude this section we remark that the same procedure can be applied to the negative discrete series of IR. Moreover, from the finite difference equation and the differentiation formulae (2), (14) and (16) we can obtain some new recurrence relations for the CGCs of the $SU_q(1,1)$ quantum algebra.

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Appendix. Calculation of the main data of the dual Hahn q-polynomials in the non-uniform lattice $x(s) = [s]_q[s+1]_q$

The dual Hahn q-polynomials are the polynomial solution of the second-order finite difference equation of the hypergeometric type on the non-uniform lattice $x(s)=[s]_q[s+1]_q$:

$$\frac{q^{s+c+a-b+2}[s-a]_q[s+b]_q[s-c]_q}{[2s+1]_q}\Delta\left[\frac{\nabla W_n^{(c)}(s,a,b)_q}{[2s-2]_q}\right] \\ +\{-[s]_q[s+1]_q+q^{c-b+1}[c]_q[b]_q+q^{a+c-b+1}[a+1]_q[b-c-1]_q\} \\ \times\frac{\Delta W_n^{(c)}(s,a,b)_q}{[2s]_q}+q^{-n+1}[n]_q W_n^{(c)}(s,a,b)_q=0. \qquad (35)$$

Since $x(-s-1)=x(s)$ and $\Delta x(s-\frac{1}{2})=-\Delta x(t-\frac{1}{2})|_{t=-s-1}$, the coefficient $\tau(s)$ in (35) is completely determined by the formula ([24, equation (3.5.3), p 75])

$$\tau(s)=\frac{\sigma(-s-1)-\sigma(s)}{\Delta x(s-\frac{1}{2})}.$$

The k-order finite difference derivative of the polynomials $W_n^{(c)}(s,a,b)_q$ is defined as

$$v_{kn}(s)=\frac{\Delta}{\Delta x_{k-1}(s)}\frac{\Delta}{\Delta x_{k-2}(s)}\cdots\frac{\Delta}{\Delta x(s)}[W_n^{(c)}(s,a,b)_q]\equiv\Delta^{(k)}[W_n^{(c)}(s,a,b)_q]$$

where $x_m(s)=x(s+m/2)$, and satisfies the following equation of the same type

$$\frac{q^{s+c+a-b+2}[s-a]_q[s+b]_q[s-c]_q}{[2s+1-k]_q}\Delta\left[\frac{\nabla v_{kn}(s)}{[2s-k-2]_q}\right]+\tau_n(s)\frac{\Delta v_{kn}(s)}{[2s-k]_q}+\mu_k v_{kn}(s)=0 \qquad (36)$$

(see [24, p 62, equation (3.1.19)]). Furthermore, we have

$$\tau_k(s)=\frac{\sigma(s+k)-\sigma(s)+\tau(s+k)\Delta x(s+m-\frac{1}{2})}{\Delta x_{k-1}(s)}$$

$$\mu_k=q^{-n+1}[n]_q+\sum_{m=0}^{k-1}\frac{\Delta\tau_m(x)}{\Delta x_m(x)}.$$

Thus, as a result, we obtain

$$\tau_k(s)=-q^{2k}[s+\tfrac{k}{2}]_q[s+\tfrac{k}{2}+1]_q+q^{c-b+k+1}[c+\tfrac{k}{2}]_q[b-\tfrac{k}{2}]_q \\ +q^{a+c-b+1-\frac{k}{2}}[a+\tfrac{k}{2}+1]_q[b-c-k-1]_q. \qquad (37)$$

The solution of the Pearson-type finite difference equation

$$\frac{\Delta}{\Delta x(s-\frac{1}{2})}[\sigma(x)\rho(x)]=\tau(x)\rho(x)$$

gives the weight function $\rho(s)$,

$$\rho(s)=\frac{q^{-s(s+1)}[s+a]_q![s+c]_q!}{[s-a]_q![s-c]_q![s+b]_q![b-s-1]_q!}.$$

Using the definition $\rho_n(s)=\rho(n+s)\prod_{k=1}^n\sigma(s+k)$ (see (5)) we obtain

$$\rho_n(s)=\frac{q^{-s(s+1+n)-n^2/2+n(a+c-b+\frac{3}{2})}[s+a+n]_q![s+c+n]_q!}{[s-a]_q![s-c]_q![s+b]_q![b-s-n-1]_q!}. \qquad (38)$$

Let us find the squared normalization factor for the dual Hahn q-polynomials. Firstly,we use the formula ([24, section 3.2.2, p 64, equation (3.7.15)])

$$d_n^2=q^{-\frac{3}{2}n^2+\frac{3}{2}n}[n]_q!B_n^2S_n \qquad (39)$$

where $B_n = (-1)^n/[n]_q!$ and S_n is a sum

$$\sum_{s_i=a}^{b-n-1} \rho_n(s_i)\Delta x_n(s_i - \tfrac{1}{2}). \tag{40}$$

To calculate it we will use the identity ($N = b - a - 1 \in \mathbb{N}$)

$$S_n = \frac{S_n}{S_{n+1}}\frac{S_{n+1}}{S_{n+2}}\cdots\frac{S_{N-2}}{S_{N-1}}S_{N-1}. \tag{41}$$

From (38) and (40) we find

$$S_{N-1} = \frac{[a+c+N-1]_q!}{[a-c]_q!}q^{-a^2-aN-(N-1)^2/2+(N-1)(a+c-b+\frac{3}{2})}.$$

To obtain S_n we will follow [24, pp 105, 106]. Using the formulae given on the mentioned pages, we find that $S_n/S_{n+1} = 1/\sigma(x^*_{n-1})$, where x^*_{n-1} is the solution of the equation $\tau_{n-1}(x^*_{n-1}) = 0$. Some straightforward but tedious algebra gives the following expression:

$$\sigma(x^*_{n-1}) = q^{-2a+2c-2b+n+2}[a+c+n]_q[b-a-n]_q[b-c-n]_q.$$

Now, collecting the expressions (39)–(41), we find that the squared normalization factor for the dual Hahn q-polynomials is equal to

$$d_n^2 = q^{-ab-bc+ac+a+c-b+1+2n(a+c-b)-n^2+5n}\frac{[a+c+n]_q!}{[n]_q![b-c-n-1]_q![b-a-n-1]_q!}.$$

To obtain the leading coefficient a_n of the polynomial and the coefficients α_n, β_n, and γ_n of the three-term recurrence relation (4) we use [24, equation (3.7.2), p 100] and formulae

$$\alpha_n = \frac{a_n}{a_{n+1}} \qquad \gamma_n = \frac{a_{n-1}}{a_n}\frac{d_n^2}{d_{n-1}^2}$$

$$\beta_n = -\frac{\alpha_n W^{(c)}_{n+1}(a,a,b)_q + \gamma_n W^{(c)}_{n-1}(a,a,b)_q}{W^{(c)}_n(a,a,b)_q} - x(a).$$

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